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On the validity of the Lorentz–Dirac equation

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Abstract

Ten conserved quantities corresponding to the symmetry of the composite system of point-like charged particle and electromagnetic field under the Poincaré group are expressed in terms of particle variables. It is shown that the Lorentz–Dirac equation contradicts the differential consequence of the 'centre-of-mass' conserved quantity which arises from the invariance of the system under Lorentz transformation.

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1. Introduction

The Lorentz–Dirac equation is an equation of motion for a charged particle under the influence of an external force as well as its own electromagnetic field. The particle's world line is described by the functions $z^\alpha(\tau)$ which give the particle's coordinates as functions of proper time τ. We define $u^\alpha(\tau) = \mathrm{d}z^\alpha(\tau)/\mathrm{d}\tau$ as the 4-velocity and $a^\alpha(\tau) = \mathrm{d}u^\alpha(\tau)/\mathrm{d}\tau$ as the 4-acceleration. The Lorentz–Dirac equation is written as

$$ma^\alpha = F^\alpha_{\text{ext}} + \tfrac{2}{3}e^2\left(\dot{a}^\alpha - u^\alpha\left(a_\mu a^\mu\right)\right) \tag{1.1}$$

where m is the particle's rest mass, e its charge, F^α_{ext} the external force and $\dot{a}^\alpha(\tau) = \mathrm{d}a^\alpha(\tau)/\mathrm{d}\tau$. The third term takes into account the energy loss due to radiation, the second one follows from a proper relativistic treatment given first by Schott (1915), and is called the 'Schott term'. If the first term is the Lorentz force, the Schott term is necessary in order to preserve the equality $u_\mu u^\mu = -1$.

The problems of runaway solutions (where acceleration grows expotentially with time) and pre-acceleration (when acceleration begins to increase prior to the time at which the external force switches on) occur in this theory [1] . They cast serious doubt on the validity of the Lorentz–Dirac equation.

The law of conservation of the total 4-momentum of a composite (particle plus field) system provides the foundation for Dirac's derivation [2] of the radiation-reaction force. The verification of energy conservation is not a trivial matter, since the Lorentz–Dirac equation is derived with the help of a mass renormalization procedure, which involves the manipulation of the divergent self-energy of a point charge.

There are many derivations which are patterned after Dirac's classical paper [2] (see, for instance, [1, 3, 4]). Although they differ from it in their technical aspects, all the derivations involve the Taylor expansion of a finite sized charged sphere in which the first two terms lead to the electromagnetic self-energy and the Abraham radiation-reaction 4-vector, respectively. Following [2], the authors enclose a world line within a thin world tube and calculate an electromagnetic flow across this surface per unit proper time. In fact, they calculate the time derivative of the energy–momentum 4-vector. My main objective is to calculate how much electromagnetic field momentum flows across hyperplane $\Sigma_t = \left\{ y \in \mathbb{M}_4 : y^0 = t \right\}$ at fixed instant time t. Thanks to such a computation we make sense of the so-called 'mass renormalization' procedure and the separation of the 'structure-independent' Schott term. These methods are very important to obtain the Lorentz–Dirac equation (1.1).

The physical meaning of a decomposition of the electromagnetic field's stress–energy tensor into radiative and bound components will be fully elucidated too. The Schott term in the Lorentz–Dirac equation originates from the bound component of the Maxwell energy–momentum tensor density. Teitelboim shows [3] that due to volume integration of this component we obtain an electromagnetic 4-momentum carried by the particle around it. In fact, by this he means the particle 4-momentum

$$p^\mu = mu^\mu - \tfrac{2}{3}e^2 a^\mu \tag{1.2}$$

which contains, apart from the usual velocity term, also a contribution from the acceleration when the particle is charged. We substantiate Teitelboim's concept so far as the particle's electromagnetic 'fur' is concerned.

The main goal of the present paper is to check the consistency of the Lorentz–Dirac equation with fundamental principles such as energy–momentum conservation and the conservation of total angular momentum. By 'fundamental principles' we mean the ten conserved quantities corresponding to Poincaré-invariance of a composite particle plus field system.

Of course, the divergent self-energy term arises unavoidably whenever one introduces point charges in classical electrodynamics. Following [1], we assume that an intrinsic structure of a charged particle is beyond the limits of classical theory (except that its 'radius' does not vanish, though it is too small to be observed). For this reason the mass renormalization is not necessary.

Our emphasis will be on rigorous calculations and exact solutions based on standard classical electrodynamics supplemented with Rohrlich's heuristic assumptions so far as the dynamics of a single charged particle is concerned [1, sections 6.2 and 6.4].

2. Energy–momentum conservation

In this section, we check a balance between electromagnetic field momentum and mechanical momentum of an arbitrarily moving particle. We only assume that the particle is asymptotically free at the remote past and at the distant future. We suppose 'that the action of the force is reasonably limited in space–time' [1].

2.1. Preliminaries

We choose the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ for Minkowski space $\mathbb{M}_4$. We use the Heaviside–Lorentz system of units with the velocity of light $c = 1$. Summation over repeated indices is understood throughout the paper; Greek indices run from 0 to 3, and Latin indices from 1 to 3. The particle trajectory

$$\begin{aligned} \zeta &: \mathbb{R} \to \mathbb{M}_4 \\ t &\mapsto (t, z^i(t)) \end{aligned} \tag{2.1}$$

is meant as a local section of trivial bundle $(\mathbb{M}_4, i, \mathrm{R})$ where the projection

$$\begin{aligned} i &: \mathbb{M}_4 \to \mathbb{R} \\ (y^0, y^i) &\mapsto y^0 \end{aligned} \tag{2.2}$$

defines the instant form of dynamics [5].

We define $u^0 := \gamma$ and $u^i := \gamma v^i$, $v^i = \mathrm{d}z^i(t)/\mathrm{d}t$, the components of the particle's 4-velocity; its 4-acceleration is $a^\mu = \gamma\, \mathrm{d}u^\mu/\mathrm{d}t$ where the factor $\gamma := 1/\sqrt{1-v^2}$. We shall use the particle's momentarily co-moving Lorentz frame (MCLF) where the particle is momentarily at rest at the instant t. The Lorentz matrix

$$\left\| \Lambda^\alpha{}_{\alpha'} \right\| = \left(\begin{array}{c|c} \frac{1}{\sqrt{1-v^2}} & \frac{v_{l'}}{\sqrt{1-v^2}} \\ \hline \frac{v^k}{\sqrt{1-v^2}} & \delta^k{}_{l'} + \varphi(v^2) v^k v_{l'} \end{array} \right) \tag{2.3}$$

where $\varphi(v^2) = v^{-2}(\gamma - 1)$, determines the transformation to MCLF where 4-velocity $u^{\alpha'} = (1, 0, 0, 0)$ and 4-acceleration $a^{\alpha'} = (0, a^{i'})$. The components $a^{i'} = \Lambda^{i'}{}_\alpha a^\alpha$ constitute a 3-vector $\boldsymbol{a}$ which is the (non-trival) spatial part of the particle acceleration taken in MCLF.

We suppose that the components of total 4-momentum of our particle plus field system are

$$p^\nu(t) = m u^\nu(t) + P \int_{\Sigma_t} \mathrm{d}\sigma_\mu T^{\mu\nu} \tag{2.4}$$

where $\mathrm{d}\sigma_\mu$ is the vectorial surface element on a space-like hypersurface Σ_t which intersects a trajectory at the point $(t, \boldsymbol{z}(t))$. (By Σ_t we take a fibre [6] of 'instant' bundle (2.2) over $t \in \mathbb{R}$.) By $T^{\mu\nu}$ we denote the components of the Maxwell energy–momentum tensor density

$$T^{\mu\nu} = f^{\mu\lambda} f^\nu{}_\lambda - \tfrac{1}{4}\eta^{\mu\nu} f^{\kappa\lambda} f_{\kappa\lambda}. \tag{2.5}$$

The tensor has an r^{-4} singularity on a particle trajectory. In equation (2.4) capital letter P denotes the principal value of the singular integral, defined by removing from Σ_t a sphere $K(0, \varepsilon)$ around the particle and then passing to the limit $\varepsilon \to 0$.

2.2. Coordinate system

An appropriate coordinate system for flat space–time is the key to the problem. The structure of (retarded) Lienard–Wiechert potential motivates the introduction of *a coordinate system centred on an accelerated world line*. A wide class of such coordinate systems was considered by Newman and Unti [7]. The set of curvilinear coordinates for flat space–time $\mathbb{M}_4$ involves the retarded time, say u, and the retarded distance r. The former is the root of the algebraic equation

$$(y^0 - u)^2 = \sum_i (y^i - z^i(u))^2 \tag{2.6}$$

which is related to the observation time t by the causality condition $t - u > 0$. The latter is the distance between an observer event y and the particle, as measured at the retarded time in the MCLF:

$$r(y) = -\eta_{\alpha\beta}(y^\alpha - z^\alpha(u)) u^\beta(u). \tag{2.7}$$

We start with the coordinate transformation

$$y^0 = u + r\Lambda^0{}_{\alpha'} n^{\alpha'} \qquad y^i = z^i(u) + r\Lambda^i{}_{\alpha'} n^{\alpha'} \tag{2.8}$$

which is a specific example of the Newman and Unti class of coordinate systems, presented in [4]. The null vector $n := (1, \boldsymbol{n})$ has the components $(1, \cos\phi \sin\vartheta, \sin\phi \sin\vartheta, \cos\vartheta)$; ϑ and ϕ are two polar angles.

To adopt these curvilinear coordinates to the instant form of dynamics (2.2), we replace the retarded distance r by the expression

$$r = \frac{\sqrt{1-v^2}}{1+(\boldsymbol{v}\cdot\boldsymbol{n})}(t-u) \tag{2.9}$$

where t is the observation time. On rearrangement, the final coordinate transformation $(y^\alpha) \mapsto (t, u, \vartheta, \phi)$ looks as follows:

$$y^0 = t \qquad y^i = z^i(u) + \frac{\sqrt{1-v^2}}{1+(\boldsymbol{v}\cdot\boldsymbol{n})}(t-u)\Lambda^i{}_{\alpha'} n^{\alpha'}. \tag{2.10}$$

Since the bundle (2.2) is trivial [6], we consider space–time $\mathbb{M}_4$ as a disjoint union of fibres $i^{-1}(t) := \Sigma_t$ parametrized by the coordinates (u, ϑ, ϕ). This coordinate system is global because different Σ's do not intersect.

2.3. The electromagnetic field's stress–energy tensor

The components of the Lienard–Wiechert potential $\hat{A} = A_\alpha \,\mathrm{d}y^\alpha$ depend on the state of the particle's motion at the retarded time only:

$$A_\alpha = e\frac{u_\alpha(u)}{r(y)}. \tag{2.11}$$

Here $u_\alpha(u)$ are the components of the velocity one-form $\hat{u}$. The electromagnetic field is written as [4]

$$\hat{f} = \frac{e}{r^2}[\hat{u} + r(a_k\hat{u} + \hat{a})] \wedge \hat{k} \tag{2.12}$$

where one-form $\hat{k} = k_\alpha \,\mathrm{d}y^\alpha$ has the components $k_\alpha = \eta_{\alpha\beta} k^\beta, k^\beta = \Lambda^\beta{}_{\beta'} n^\beta$, and scalar $a_k = k_\alpha a^\alpha$. To express the components $f_{\alpha\beta}$ in terms of curvilinear coordinates (2.10) we substitute the right-hand side of equation (2.9) for the retarded distance r in this expression.

It is straightforward to substitute these components into equation (2.5) to calculate the electromagnetic field's stress–energy tensor. Following [3], we present $T^{\alpha\beta}$ as a sum of radiative and bound components,

$$T^{\alpha\beta} = T^{\alpha\beta}_{\mathrm{rad}} + T^{\alpha\beta}_{\mathrm{bnd}} \tag{2.13}$$

where

$$\begin{aligned} 4\pi T^{00}_{\mathrm{rad}} &= \frac{e^2}{(t-u)^2}\frac{[1+(\boldsymbol{v}\cdot\boldsymbol{n})]^4}{(1-v^2)^2}(\boldsymbol{a}^2 - (\boldsymbol{a}\cdot\boldsymbol{n})^2) \\ 4\pi T^{0i}_{\mathrm{rad}} &= \frac{e^2}{(t-u)^2}\frac{[1+(\boldsymbol{v}\cdot\boldsymbol{n})]^3}{(1-v^2)^{3/2}}(\boldsymbol{a}^2 - (\boldsymbol{a}\cdot\boldsymbol{n})^2)(v^{i'} + n^{i'})\Lambda^i{}_{i'} \end{aligned} \tag{2.14}$$

are the radiative components, and

$$\begin{aligned} 4\pi T^{00}_{\mathrm{bnd}} = \frac{1}{2}\frac{e^2}{(t-u)^4}&\frac{[1+(\boldsymbol{v}\cdot\boldsymbol{n})]^4}{(1-v^2)^3}[1 - 2(\boldsymbol{v}\cdot\boldsymbol{n})^2 + v^2] \\ &+ 2\frac{e^2}{(t-u)^3}\frac{[1+(\boldsymbol{v}\cdot\boldsymbol{n})]^4}{(1-v^2)^{5/2}}[(\boldsymbol{a}\cdot\boldsymbol{v}) - (\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n})] \end{aligned} \tag{2.15}$$

$$4\pi T^{0i}_{\rm bnd} = \frac{e^2}{(t-u)^4}\frac{[1+(\boldsymbol{v}\cdot\boldsymbol{n})]^4}{(1-v^2)^{5/2}}[v^{i'} - (\boldsymbol{v}\cdot\boldsymbol{n})n^{i'}]\Lambda^i{}_{i'} + \frac{e^2}{(t-u)^3}\frac{[1+(\boldsymbol{v}\cdot\boldsymbol{n})]^3}{(1-v^2)^2}([(\boldsymbol{a}\cdot\boldsymbol{v}) - (\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n})]v^{i'} + [(\boldsymbol{a}\cdot\boldsymbol{v}) - (\boldsymbol{a}\cdot\boldsymbol{n}) - 2(\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{a}\cdot\boldsymbol{n})]n^{i'} + [1+(\boldsymbol{v}\cdot\boldsymbol{n})]a^{i'})\Lambda^i{}_{i'} \tag{2.16}$$

are the bound components. The results coincide with the components $T^{0\alpha}$ obtained in [4, equations (5.4) and (5.5)] where k^α should be replaced by $\Lambda^\alpha{}_{\alpha'}n^{\alpha'}$ and the right-hand side of equation (2.9) should be substituted for the retarded distance.

2.4. *Electromagnetic field momentum*

Now we calculate the electromagnetic field momentum

$$p^\mu_{\rm em} = \int_{\Sigma_t} {\rm d}\sigma_0 T^{0\mu} \tag{2.17}$$

where an integration hypersurface $\Sigma_t = \left\{y \in \mathbb{M}_4 : y^0 = t\right\}$ is a surface of constant t.

The surface element is given by ${\rm d}\sigma_0 = \sqrt{-g}\,{\rm d}u\,{\rm d}\vartheta\,{\rm d}\phi$ where

$$\sqrt{-g} = \frac{(1-v^2)^2}{[1+(\boldsymbol{v}\cdot\boldsymbol{n})]^3}(t-u)^2\sin\vartheta \tag{2.18}$$

is the determinant of the metric tensor of Minkowski space viewed in curvilinear coordinates (2.10). The angular integration can be handled via the relations

$$\begin{aligned} \int_0^\pi {\rm d}\vartheta\,\sin\vartheta \int_0^{2\pi} {\rm d}\phi\, n^i &= 0 \\ \int_0^\pi {\rm d}\vartheta\,\sin\vartheta \int_0^{2\pi} {\rm d}\phi\, n^i\,n^j &= \frac{4\pi}{3}\delta^{ij} \\ \int_0^\pi {\rm d}\vartheta\,\sin\vartheta \int_0^{2\pi} {\rm d}\phi\, n^i\,n^j\,n^k &= 0. \end{aligned} \tag{2.19}$$

The calculation reveals that the decomposition of the stress–energy tensor into radiative and bound components is meaningful. Indeed, the radiative component (2.14) scales as r^{-2}; its contribution is regular:

$$\begin{aligned} p^0_{\rm rad} &= \int_{y^0=t} {\rm d}\sigma_0 T^{00}_{\rm rad} = \frac{2}{3}e^2\int_{-\infty}^t {\rm d}u\,\boldsymbol{a}^2(u) \\ p^i_{\rm rad} &= \int_{y^0=t} {\rm d}\sigma_0 T^{0i}_{\rm rad} = \frac{2}{3}e^2\int_{-\infty}^t {\rm d}u\,\boldsymbol{a}^2(u)v^i(u). \end{aligned} \tag{2.20}$$

The radiative momentum is accumulated: its amount in Σ_t at fixed time t depends on all previous motion of a source. While the bound 4-momentum depends on the state of the particle's motion at the observation time only! The matter is that the total (retarded) time derivatives arise from angular integration:

$$\begin{aligned} p^0_{\rm bnd} = P\int_{y^0=t} {\rm d}\sigma_0 T^{00}_{\rm bnd} &= \frac{2}{3}e^2\int_{-\infty}^t {\rm d}u\left[\frac{1}{(t-u)^2}\left(-\frac{1}{4}+\frac{1}{1-v^2}\right) + \frac{1}{t-u}\frac{2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})}{(1-v^2)^2}\right] \\ &= \frac{2}{3}e^2\lim_{u\to t}\left(-\frac{1}{4}+\frac{1}{1-v^2(u)}\right)\frac{1}{t-u} \end{aligned} \tag{2.21}$$

$$p^i_{\rm bnd} = P\int_{y^0=t} {\rm d}\sigma_0 T^{0i}_{\rm bnd} = \frac{2}{3}e^2\int_{-\infty}^t {\rm d}u\left[\frac{1}{(t-u)^2}\frac{v^i}{1-v^2} + \frac{1}{t-u}\left(\frac{\dot v^i}{1-v^2} + \frac{2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})v^i}{(1-v^2)^2}\right)\right]$$
$$= \frac{2}{3}e^2\lim_{u\to t}\frac{v^i(u)}{1-v^2(u)}\frac{1}{t-u}. \qquad (2.22)$$

This is explained by Teitelboim in [3, p 1581] 'It is of interest to emphasize that the tensor $T^{\mu\nu}_{\rm bnd}$ and, in particular, its components $T^{0\mu}_{\rm bnd}$, which are to be interpreted as the negatives of the energy and momentum densities in the rest frame, are retarded functions. Thus a change in the energy–momentum density on $\sigma(\tau)$ can be caused only by a change of the kinematics of the charge prior to τ[1]. Nevertheless, if one adds all the contributions from the various volume elements, the net result depends only on a neighbourhood of the present event $z(\tau)$. Thus it looks as if the charge carried a rigid electromagnetic cloud, but a truly rigid eletromagnetic configuration would contradict the finite speed of propagation of the interactions.'

From the formal point of view the bound components (2.21) and (2.22), involved in particle 4-momentum, are divergent. We arrive at the gap between structureless point particles and finite field energies. In Rohrlich's opinion [1], it is impossible to fill in the gap using the methods of classical electrodynamics. A higher level theory is necessary. For this reason we do not make any assumptions about the particle structure, its charge distribution and its size. We only assume that the particle 4-momentum is finite. To substantiate our point of view we are going to analyse commonly used manipulations with divergent terms (2.21) and (2.22).

2.5. Schott term

We face the problem of how the Schott term arises due to integration of the bound component of the energy–stress tensor. One usually works in the frame of the covariant approach where the proper time τ is used as an evolution parameter. Since ${\rm d}\tau = \sqrt{1-v^2(t)}\,{\rm d}t$, we substitute the small parameter ε for $\sqrt{1-v^2(t)}(t-u)$ in equations (2.21) and (2.22). In terms of covariant coordinates the components of the singular 4-momentum involve the term

$$\frac{2}{3}e^2\lim_{\varepsilon\to 0}\frac{u^\mu(\tau-\varepsilon)}{\varepsilon}. \qquad (2.23)$$

(Only the zeroth component has the additional term.) We are interested in the limit $\varepsilon\to 0$ and, therefore, we expand this singularity in the immediate vicinity of the world line. In the Taylor expansion of equation (2.23) the structureless term is proportional to the particle 4-acceleration.[2] It is the well-known Schott term involved in the Lorentz–Dirac equation (1.1).

2.6. Renormalization of mass

It is often assumed that the particle is a 'matter' core 'dressed' in the electromagnetic 'cloud'. The divergent term—the first term of the Taylor expansion of (2.23)—should be added to a rest mass of the 'matter' core, so that this already renormalized mass is meaningful.

We have a problem of how such a renormalization procedure for bound 4-momentum with components (2.21) and (2.22) should be defined. Indeed, the zeroth component contains a term which is not proportional to the zeroth component of 4-velocity while the spatial components are proportional to u^i. The reason is that we use surface $\Sigma_t = \left\{y\in\mathbb{M}_4 : y^0 = t\right\}$ as an

[1] The author deals with covariant proper time τ; $\sigma(\tau)$ is the spacelike surface (2.24) which intersects a world line at point $z(\tau) = (t, \boldsymbol{z}(t))$.

[2] One usually assumes some radius of the particle and proclaims the structure-independent terms as those of true physical meaning.

integration hypersurface in equation (2.17). Rohrlich [1] and Teitelboim [3] suggest that the momentarily co-moving Lorentz frame of the charge plays a privileged role in the definition of the energy–momentum corresponding to the bound part of the energy–momentum tensor. The authors use the spacelike surface σ_t defined by

$$u_\mu(\tau)(y^\mu - z^\mu(\tau)) = 0 \tag{2.24}$$

as the integration hypersurface. Our aim is to make strict sense of this ‘privileged role’.

So, we have to calculate the volume integral (2.17) over tilted hyperplanes. To apply our previous results we make such a Lorentz transformation Ω that a tilted hyperplane becomes $\Sigma_{t'} = \left\{y \in \mathbb{M}_4 : y^{0'} = t'\right\}$. After trivial calculations we arrive at

$$\begin{aligned} p^\mu_{\rm bnd} &= \int_{\sigma_t} {\rm d}\sigma_\nu T^{\nu\mu}_{\rm bnd} \\ &= \int_{y^{0'}=t'} {\rm d}\sigma_{0'} T^{0'\alpha'}_{\rm bnd} \Omega_{\alpha'}{}^\mu \\ &= \Omega^\mu{}_{\alpha'} p^{\alpha'}_{\rm bnd}. \end{aligned} \tag{2.25}$$

Using $\Omega^\mu{}_{\alpha'} = \Lambda^\mu{}_{\alpha'}$, where matrix elements $\Lambda^\mu{}_{\alpha'}$ are given by equation (2.3), we arrive at the frame in which the particle is momentarily at rest at time t. In MCLF the particle velocity $u' = (1, 0, 0, 0)$ and the spatial components (2.22) of bound 4-momentum vanish:

$$p^{0'}_{\rm bnd} = \lim_{u'\to t'} \frac{1}{2} e^2 \frac{1}{t' - u'} \qquad p^{i'}_{\rm bnd} = 0. \tag{2.26}$$

As usual, the divergent quantity $e^2/2\varepsilon$ is linked with the mechanical ‘matter’ mass of a particle, so that the renormalized mass is considered to be finite.

We see that the computation of the rate of the electromagnetic field momentum which flows across all the hyperplane $y^0 = {\rm const}$ does not contradict the usual approach in which one calculates an electromagnetic flow across a thin tube around the world line per unit proper time. But it allows us to explain the meaning of manipulations with divergent terms such as ‘renormalization’ of mass and separation of the ‘structure-independent’ Schott term.

3. Total angular momentum tensor of the electromagnetic field

The charged particle cannot be separated from its bound electromagnetic ‘cloud’. We would like to construct the particle 4-momentum in terms of its state functions (velocity, acceleration, etc). The usual approach based on the ‘renormalization’ of mass and separation of the ‘structure-independent’ Schott term leads to Teitelboim’s formula (1.2). This approach is mathematically incorrect. To obtain additional information we calculate the conserved quantities corresponding to the invariance of the theory under proper homogeneous Lorentz transformations.

We are now concerned with the total angular momentum tensor of the electromagnetic field [1]:

$$M^{\mu\nu}_{\rm em} = \int_{\Sigma_t} {\rm d}\sigma_0 (y^\mu T^{0\nu} - y^\nu T^{0\mu}). \tag{3.1}$$

Conservation of the space part $M^{ij}_{\rm em}$ of the tensor $M^{\mu\nu}_{\rm em}$ is due to invariance under space rotations. Conservation of the mixed space–time components, $M^{0i}_{\rm em}$, expresses the centre-of-mass theorem. It takes place due to invariance under Lorentz transformations.

We substitute equations (2.13) and (2.10) into equation (3.1) to calculate the electromagnetic field’s angular momentum tensor. Routine scrupulous calculation reveals the (divergent) components of bound 4-momentum (2.21) and (2.22) in the proper places! The components of the angular momentum tensor are as follows:

$$J^k_{\rm em} := \varepsilon^k{}_{ij}\, M^{ij}_{\rm em} = \varepsilon^k{}_{ij}\, z^i(t) p^j_{\rm bnd} + \frac{2}{3}e^2 \int_{-\infty}^t {\rm d}u \boldsymbol{a}^2(u) \varepsilon^k{}_{ij}\, z^i(u) v^j(u) + \frac{2}{3}e^2 \int_{-\infty}^t {\rm d}u\, \varepsilon^k{}_{ij}\, v^i(u) a^j(u) \tag{3.2}$$

$$K^i_{\rm em} := -M^{0i}_{\rm em} = -t p^i_{\rm bnd} + z^i(t) p^0_{\rm bnd} + \frac{2}{3}e^2 \int_{-\infty}^t {\rm d}u \boldsymbol{a}^2(u)[z^i(u) - v^i(u)u] + \frac{4}{3}e^2 \int_{-\infty}^t {\rm d}u \frac{v^i(u)(\boldsymbol{a}\cdot\boldsymbol{v})}{\sqrt{1-\boldsymbol{v}^2}}. \tag{3.3}$$

This result reinforces our conviction that the bound momentum and its 'matter' mechanical counterpart constitute the 4-momentum $p_{\rm part}$ of the charged structureless particle.

Taking into account the mechanical part of the angular 4-momentum, we obtain the following ten conserved quantities which are due to the invariance of our composite particle plus field system under infinitesimal transformations of the Poincaré group:

$$p^0 = p^0_{\rm part} + \frac{2}{3}e^2 \int_{-\infty}^t {\rm d}u \boldsymbol{a}^2(u) \tag{3.4}$$

$$p^i = p^i_{\rm part} + \frac{2}{3}e^2 \int_{-\infty}^t {\rm d}u \boldsymbol{a}^2(u) v^i(u) \tag{3.5}$$

$$J^k = \varepsilon^k{}_{ij}\, z^i(t) p^j_{\rm part} + \frac{2}{3}e^2 \int_{-\infty}^t {\rm d}u \boldsymbol{a}^2(u) \varepsilon^k{}_{ij}\, z^i(u) v^j(u) + \frac{2}{3}e^2 \int_{-\infty}^t {\rm d}u\, \varepsilon^k{}_{ij}\, v^i(u) a^j(u) \tag{3.6}$$

$$K^i = -t p^i_{\rm part} + z^i(t) p^0_{\rm part} + \frac{2}{3}e^2 \int_{-\infty}^t {\rm d}u \boldsymbol{a}^2(u)[z^i(u) - v^i(u)u] + \frac{4}{3}e^2 \int_{-\infty}^t {\rm d}u \frac{v^i(u)(\boldsymbol{a}\cdot\boldsymbol{v})}{\sqrt{1-\boldsymbol{v}^2}}. \tag{3.7}$$

Thus we finally arrive at the natural decomposition of the conserved quantities into the particle component and radiative component. The former depends on the instant characteristics of the charged particle while the latter is accumulated with time.

To construct the particle motion equation we only need to consider the vicinity of the world line. We calculate how much electromagnetic field momentum and angular momentum flow across hypersurface Σ_t. We can do it at a time $t + \Delta t$. We demand that the change in these quantities be balanced by a corresponding change in those of the particle, so that the total energy–momentum $(p^0, \boldsymbol{p})$ and angular momentum $(\boldsymbol{J}, \boldsymbol{K})$ are properly conserved. Via the differentiation of equations (3.4)–(3.7) we arrive at the following system of differential equations:

$$\dot p^0_{\rm part} = -\tfrac{2}{3}e^2 \boldsymbol{a}^2(t) \tag{3.8}$$

$$\dot p^i_{\rm part} = -\tfrac{2}{3}e^2 \boldsymbol{a}^2(t) v^i(t) \tag{3.9}$$

$$\varepsilon^k{}_{ij}\, v^i(t) p^j_{\rm part} = -\tfrac{2}{3}e^2 \varepsilon^k{}_{ij}\, v^i(t) a^j(t) \tag{3.10}$$

$$p^i_{\rm part} - v^i(t) p^0_{\rm part} = \tfrac{4}{3}e^2 \frac{v^i(t)(\boldsymbol{a}\cdot\boldsymbol{v})}{\sqrt{1-\boldsymbol{v}^2}}. \tag{3.11}$$

Its solution is a motion with constant velocity where $p^\mu_{\rm part}$ do not change.

The problem of particle motion in the presence of an external force requires careful consideration. We do not know the rate of the external device in the balance condition of the total angular momentum $(\boldsymbol{J}, \boldsymbol{K})$. (Considering the energy–momentum we use the Lorentz force, or capacity for a non-electromagnetic force.)

Of course, one would prefer an expression which explains how 4-momentum of charged particle depends on its velocity and acceleration etc. It is obvious that this expression should satisfy the differential consequences of the total angular momentum. To check Teitelboim's expression we substitute the right-hand side of equation (1.2) for $p_{\rm part}$ in equations (3.10) and (3.11). We see that equation (3.10) is satisfied identically while equation (3.11) is not fulfilled. Therefore, Teitelboim's expression (1.2) contradicts the differential consequence of 'centre-of-mass' conserved quantity.

4. Conclusions

We can briefly summarize our conclusions as follows:

- a charged particle cannot be separated from its bound electromagnetic 'fur', so that the 4-momentum of the particle is the sum of the mechanical momentum and the electromagnetic bound 4-momentum.
- Teitelboim's expression for particle 4-momentum as a linear function of particle's velocity and acceleration contradicts the structure of centre-of-mass conserved quantity originated from an invariance of our composite system under Lorentz transformations.

Moreover, the system of six linear equations (3.10) and (3.11) in variables $p^{\mu}_{\rm part}$ does not possess a solution whenever the particle's motion is accelerated. Does it mean that there is no expression of type (1.2) within an interaction area? The problem requires careful consideration. It is worth noting that in the absence of an external force the motion of a classical point charge satisfies the law of inertia (Newton's first law).

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